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LETTER TO THE EDITOR

Survival of a diffusing particle in a transverse shear flow: a first-passage problem with continuously varying persistence exponent

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Abstract

We consider a particle diffusing in the y-direction, $dy/dt = \eta(t)$, subject to a transverse shear flow in the x-direction, $dx/dt = f(y)$, where $x \geqslant 0$ and $x = 0$ is an absorbing boundary. We treat the class of models defined by $f(y) = \pm v_\pm(\pm y)^\alpha$ where the upper (lower) sign refers to $y > 0$ ($y < 0$). We show that the particle survives with probability $Q(t) \sim t^{-\theta}$ with $\theta = 1/4$, independent of α, if $v_+ = v_-$. If $v_+ \neq v_-$, however, we show that θ depends on both α and the ratio v_+/v_-, and we determine this dependence.

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First passage problems are an important aspect of the theory of stochastic processes. Applications include population dynamics, chemical reactions and many other problems in science [1]. This class of problems has attracted a resurgence of interest in the last decade. The number of exactly-solved models, however, is still quite small [1, 2]. The classic example is a one-dimensional random walker, $dx/dt = \eta(t)$, where $\eta(t)$ is a Gaussian white noise with mean zero and correlator $\langle\eta(t)\eta(t')\rangle = 2D\delta(t - t')$. If the particle starts at $x > 0$, and there is an absorbing boundary at $x = 0$, the particle survives at time t with probability $Q(x, t) = \mathrm{erf}(x/\sqrt{4Dt})$ [1]. For large t this gives $Q(x, t) \sim x/(Dt)^{1/2}$, defining an exponent $\theta = 1/2$ (often called a 'persistence exponent' [2]) through the relation $Q(x, t) \sim t^{-\theta}$ for large t.

In this letter, we introduce a new class of models for which θ can be determined exactly and has, in general, a non-trivial value. We treat a simple model of a particle diffusing in the y-direction, $dy/dt = \eta(t)$, but subject to a transverse flow field ('shear') in the x-direction, $dx/dt = f(y)$, with an absorbing boundary at $x = 0$. We show that, in general, the exponent θ depends continuously on the model parameters, except when the function $f(y)$ is an odd function, $f(-y) = -f(y)$, when we find $\theta = 1/4$. Two specific examples with odd $f(y)$ have attracted some attention in the literature. The first, $f(y) = y$, is equivalent to $d^2x/dt^2 = \eta(t)$, the 'random acceleration model', for which the survival probability is known rigorously to decay as $Q(t) \sim t^{-1/4}$ [3]. The second model for which we are aware of previous work is the case $f(y) = v\,\mathrm{sgn}(y)$. Redner and Krapivsky [4] have presented qualitative arguments that $\theta = 1/4$ for this model too, and indeed have conjectured that $\theta = 1/4$ for all odd functions $f(y)$. Below we present an argument, based on the Sparre Andersen theorem [5], that this conjecture is correct. First, however, we derive the result analytically for the class of odd functions $f(y) = v\,\mathrm{sgn}(y)|y|^\alpha$. Furthermore, we will show that when the amplitude v takes different values, v_+ and v_-, for $y > 0$ and $y < 0$ respectively, the exponent θ becomes a non-trivial function, whose form we determine explicitly, of the exponent α and the ratio v_+/v_-. Our result thus greatly extends the class of first-passage problems for which the value of θ is known exactly.

The starting point of the calculation is the two equations

$$\dot{y} = \eta(t) \tag{1}$$

$$\dot{x} = f(y) = v_\pm\,\mathrm{sgn}(y)|y|^\alpha \tag{2}$$

where the upper (lower) sign refers to $y > 0$ ($y < 0$), dots indicate time derivatives and $\eta(t)$ is a Gaussian white noise. The final result for θ can be written in the form

$$\theta = \frac{1}{4} - \frac{1}{2\pi\beta}\tan^{-1}\left[\frac{v_+^\beta - v_-^\beta}{v_+^\beta + v_-^\beta}\tan\left(\frac{\pi\beta}{2}\right)\right] \tag{3}$$

where $\beta = 1/(2+\alpha)$. Note that when $v_+ = v_-$ the result reduces to $\theta = 1/4$, independent of α, as claimed. The general result is easily understood in the limits $v_- \to 0$ and $v_+ \to 0$. For $v_- \to 0$, there is no mechanism by which the particle can reach the absorbing boundary at $x = 0$, so $\theta = 0$ as predicted by (3). For $v_+ \to 0$, in order to survive the particle should avoid the $y < 0$ region. The probability to remain in the $y > 0$ region decays as $t^{-1/2}$ as in the one-dimensional random walk, i.e., $\theta = 1/2$, again in accord with (3).

We now outline the derivation of equation (3). From equations (1) and (2), we can write down the backward Fokker–Planck equation

$$\frac{\partial Q}{\partial t} = D\frac{\partial^2 Q}{\partial y^2} \pm v_\pm(\pm y)^\alpha \frac{\partial Q}{\partial x} \tag{4}$$

where $Q(x, y, t)$ is the probability that the particle still survives at time t given that it started at position (x, y), and we will take $x \geqslant 0$. The absorbing boundary at $x = 0$ leads to the boundary condition $Q(0, y, t) = 0$ for all $y < 0$. The initial condition is $Q(x, y, 0) = 1$ for all $x > 0$.

Solving the full initial value problem is difficult so we will specialize to the late-time scaling regime where $Q(x, y, t) \sim t^{-\theta}$. This approach exploits a generalization of the method recently introduced by Burkhardt for $\alpha = 1$ (the random acceleration problem) [6]. Dimensional arguments give us the appropriate variables for the problem. From (1) and (2), we obtain $y \sim (Dt)^{1/2}$ and $\dot{x} \sim v(Dt)^{\alpha/2}$ giving $x \sim (v/D)y^{1/\beta}$, where $\beta = 1/(2+\alpha)$ and v carries the dimension of the amplitudes $v_\pm$. The combination $z = vy^{1/\beta}/Dx$ serves as a dimensionless scaling variable, which can also be seen directly from equation (4). This gives, for large t, $Q(x, t, y) \sim (y^2/Dt)^\theta G(z)$, where $G(z)$ is a scaling function. This is a little loose, however, as we expect different solutions for $y > 0$ and $y < 0$ due to the non-analyticity[1] of $f(y)$ at $y = 0$. Taking this into account, and expressing the prefactor of the scaling function in terms of x rather than y for convenience, we write, for $t \to \infty$,

$$Q(x, y, t) \sim \left(\frac{x^{2\beta}}{t}\right)^\theta F_\pm\left(\pm\frac{v_\pm(\pm y)^{1/\beta}}{Dx}\right) \tag{5}$$

[1] The function $f(y)$ defined by equation (2) is non-analytic at $y = 0$ except when α is an odd integer and $v_+ = v_-$. The subsequent analysis, however, includes these cases too.

where $F_{\pm}(z)$ is the scaling function for $y > 0$ (+) or $y < 0$ (−). The (dimensional) prefactors (for $y > 0$ and $y < 0$) in equation (5) have been omitted since equation (4) is linear. The functions $F_{+}(z)$ and $F_{-}(z)$ are defined such that the prefactor is the same for $y > 0$ and $y < 0$.

Inserting the form (5) into the backward Fokker–Planck equation (4), we see immediately that the term $\partial Q/\partial t$ leads to a term of order $t^{-(\theta+1)}$, which is subdominant for large t and can therefore be dropped. The remaining terms give

$$zF''_{\pm}(z) + (1-\beta-\beta^2 z)F'_{\pm}(z) + 2\beta^3\theta F_{\pm}(z) = 0. \tag{6}$$

Expressed in terms of the variable $u = \beta^2 z$, this equation becomes Kummer's equation. Independent solutions are the confluent hypergeometric functions $M(-2\beta\theta, 1-\beta, \beta^2 z)$ and $U(-2\beta\theta, 1-\beta, \beta^2 z)$ [7]. The appropriate solutions for each domain are selected from the limiting behaviour as $x \to 0$. For $y > 0$, we require $Q(0, y, t) \sim (y^2/t)^{\theta}$ which implies, from equation (5), that $F_{+}(z) \sim z^{2\beta\theta}$ for large z. The function $M(-2\beta\theta, 1-\beta, \beta^2 z)$ diverges exponentially for $z \to \infty$, so must be rejected for $y > 0$. For $y < 0$, Q must vanish at $x = 0+$ (because the flow field immediately takes the particle on to the absorbing boundary). For large negative z, both $M(-2\beta\theta, 1-\beta, \beta^2 z)$ and $U(-2\beta\theta, 1-\beta, \beta^2 z)$ behave as $(-z)^{2\beta\theta}$, so in this regime we retain both solutions and fix θ by the requirement that the coefficient of $(-z)^{2\beta\theta}$ vanishes. Thus, we write

$$F_{+}(z) = AU\left(-2\beta\theta, 1-\beta, \frac{v_{+}\beta^2 y^{1/\beta}}{Dx}\right) \tag{7}$$

$$F_{-}(z) = BU\left(-2\beta\theta, 1-\beta, \frac{-v_{-}\beta^2(-y)^{1/\beta}}{Dx}\right) + CM\left(-2\beta\theta, 1-\beta, \frac{-v_{-}\beta^2(-y)^{1/\beta}}{Dx}\right). \tag{8}$$

Relations between the coefficients A, B and C can be obtained by imposing the appropriate continuity conditions at $y = 0$. Requiring $F_{+}(0) = F_{-}(0)$ gives, using $M(a, b, 0) = 1$ and, for $b < 1$, $U(a, b, 0) = \pi[\sin(\pi b)\Gamma(b)\Gamma(1+a-b)]^{-1}$ [7],

$$C = (A-B)\frac{\pi}{\sin(\pi\beta)}\frac{1}{\Gamma[1-\beta]\Gamma[\beta(1-2\theta)]} \tag{9}$$

for all $\beta > 0$. The second relation between A, B and C is obtained by noting that, for $\alpha > -1$, equation (4) implies that the first derivative, $\partial Q/\partial y$, is continuous at $y = 0$. The term linear in y comes from a term of order z^{1-b} in the small-z expansion of $U(a, b, z)$. Since $b = 1-\beta$ here, this term is of order z^{β}, i.e., linear in y since the scaling variable z is proportional to $y^{1/\beta}$, as in equation (5). Demanding that this linear term has the same coefficient in both regimes ($y > 0$ and $y < 0$) yields

$$Av_{+}^{\beta} = Bv_{-}^{\beta}. \tag{10}$$

Finally, we need the asymptotic behaviour of $F_{-}(z)$ for $z \to -\infty$. From the asymptotic properties of the functions $M(a, b, z)$ and $U(a, b, z)$ [7] one readily obtains

$$F_{-}(z) \to \left[B\frac{\sin[\pi\beta(1-4\theta)/2]}{\sin[\pi\beta/2]} + C\frac{\Gamma[1-\beta]}{\Gamma[1-\beta(1-2\theta)]}\right](-z)^{2\beta\theta} \qquad z \to -\infty. \tag{11}$$

Since $F_{-}(z)$ should vanish for $z \to -\infty$, due to the absorbing boundary at $x = 0$, the coefficient of $(-z)^{2\beta\theta}$ must vanish in equation (11). This condition, combined with equations (9) and (10), gives the result presented in equation (3).

The case $v_{+} = v_{-}$, where $f(y)$ is an odd function, is especially simple. Equations (10) and (9) give $A = B$ and $C = 0$ respectively for this case whence, from equation (11), one obtains $\sin[\pi\beta(1-4\theta)/2] = 0$. The desired solution corresponds to the smallest positive value for θ, which is $\theta = 1/4$. Redner and Krapivsky [4] have conjectured that $\theta = 1/4$ for *all* odd functions $f(y)$. A general argument for this result can be made using the Sparre Andersen theorem [5]. The theorem deals with a one-dimensional random walk, with an absorbing boundary at $x = 0$ and step sizes that are independent, identically distributed random variables drawn from any continuous, symmetric distribution. The result of the theorem is that the probability that the walker still survives after N steps decreases as $N^{-1/2}$ for large N. To apply this result to our problem, we treat the successive crossings of the x-axis by our diffusing particle as defining the steps of the walk, i.e., successive crossings at $x = x_n$ and $x = x_{n+1}$ correspond to a step of length $x_{n+1} - x_n$ in an effective random walk along the x-axis. Clearly, the distribution of step sizes is continuous, and is symmetric for any odd $f(y)$. Since the number of crossings in time t scales as $N \sim t^{1/2}$, the survival probability decays as $Q \sim N^{-1/2} \sim t^{-1/4}$ [8][2]. If $v_+ \neq v_-$, the distribution of step sizes is no longer symmetric, the conditions of the Sparre Andersen theorem do not hold, and θ is different from 1/4. A superficially related model is the Matheron-de Marsily (MdM) model [9], introduced to model the dispersion of tracer particles in porous rocks. In this model, the function $f(y)$ in equation (2) is a *random* function of y with zero mean. The MdM model also has $\theta = 1/4$ [10], but the Sparre Anderson theorem cannot naively be applied because $f(y)$ is not odd.

Returning to our original model, we note that two special values of α are of particular interest. The case $\alpha = 0$ corresponds to the original 'windy cliff' model of Redner and Krapivsky [4], generalized to different drift velocities, v_+ and v_-, for y positive and y negative. If we transform to a frame of reference moving at speed u along the x-axis, then in the new frame the flow velocities have values $\pm v_\pm - u$, while the absorbing boundary has velocity $-u$. The expression for θ presented in equation (3) therefore applies equally well to this class of moving boundary problems. The second interesting value is the case $\alpha = 1$. Eliminating y from equations (1) and (2), one obtains a random acceleration process $\ddot{x} = v_\pm \eta(t)$, with different strength noise forces depending on the sign of the velocity.

The result (3) has been checked by computer simulations for a range of α and v_+/v_-. The results are presented in figure 1. Motion in the y-direction was modelled by a discrete-time lattice random walk, while the x coordinate is treated as a continuous variable. The initial condition was $x = 0$, $y = 0$, with the first step taken in the positive y-direction. The simulations enter the power-law regime, where $Q(t) \sim t^{-\theta}$, quite quickly, making it relatively straightforward to obtain an accurate estimate of the exponent θ. The data (symbols) are in excellent agreement with the predictions (lines) in all cases. Note that all curves and data sets cross at $\theta = 1/4$, $v_+/v_- = 1$, as anticipated.

The present exact approach is restricted to the case where the flow velocity $f(y)$ has a pure power-law form, with the *same* power, for both $y < 0$ and $y > 0$, although the amplitudes in the two regimes can be different. What can we say about the general case? The case of different powers for $y > 0$ and $y < 0$ cannot be treated analytically within the present approach because the assumed scaling form, equation (5), for $Q(x, y, t)$ does not lead to a consistent solution for both signs of y. Consider, however, the case where the function $f(y)$ has, for $y \to \pm\infty$, an asymptotic power-law form with the same power for both signs of y. Since trajectories that survive for a long time explore large values of $|y|$ [4], we conjecture that the exponent θ describing the asymptotic decay of $Q(t)$ is given by equation (3) with the exponent α and amplitude ratio v_+/v_- extracted from the limiting behaviour of $f(y)$ for $y \to \pm\infty$. This line of reasoning leads immediately to the further conjecture that when the

[2] Similar arguments have been presented for the special cases $f(y) = \mathrm{sgn}(y)$, in [4], and $f(y) = y$ in [4] and [8]. Here, we stress the generality of this approach, already hinted at in [4].

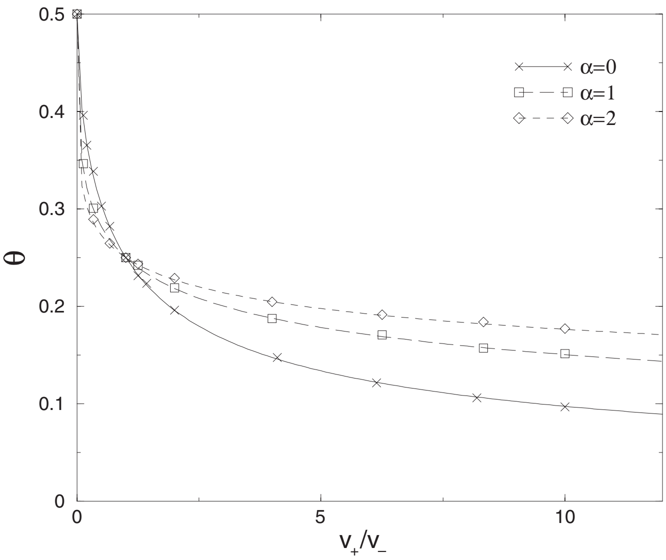

Figure 1. Simulation data for θ for a range of values of the exponent α and amplitude ratio v_+/v_- in equation (2). The continuous curves are the predictions of equation (3). The errors on the data points are smaller than the sizes of the symbols.

asymptotic behaviour of $f(y)$ for $y \to \pm\infty$ is described by *different* powers, $\alpha_\pm$, the larger power dominates the $t \to \infty$ behaviour, such that if $\alpha_+ > \alpha_-$ then $\theta = 0$ (i.e., there is a nonzero probability that the particle escapes to infinity), while if $\alpha_+ < \alpha_-$ then $\theta = 1/2$.

One may also consider the effect of diffusion in the x-direction as well as the y-direction. For the case $f(y) = v\,\mathrm{sgn}(y)$, corresponding to $\alpha = 0$, it was argued in [4] that the diffusion in the x-direction is asymptotically irrelevant compared to drift since $x_{\mathrm{diff}} \sim (Dt)^{1/2}$ whereas $x_{\mathrm{drift}} \sim vt$. Applying this reasoning to general α gives $x_{\mathrm{drift}} \sim t^{(2+\alpha)/2}$, so drift in the x-direction dominates diffusion for all $\alpha > -1$, the latter condition coinciding with the range of α for which our analytic approach is valid. We conclude that diffusion in the x-direction does not change the value of θ (for $\alpha > -1$). A more formal way to see this is to introduce a term $D_x \partial^2 Q/\partial x^2$, corresponding to diffusion in the x-direction, on the right-hand side of equation (4). Under the scale change $y \to \lambda y$, $t \to \lambda^2 t$ and $x \to \lambda^{1/\beta} x$, the equation retains its form except for a rescaled value of D_x, given by $D_x' = \lambda^{-2(1+\alpha)} D_x$. This shows that D_x scales to zero on large length and time scales and is therefore asymptotically irrelevant.

In summary, we have obtained exact results for the persistence exponent θ for a class of two-dimensional stochastic processes in which a particle diffuses in the y-direction and has a deterministic, y-dependent drift in the x-direction and an absorbing boundary at $x = 0$. The universal exponent $\theta = 1/4$ is obtained when the drift velocity is an odd function of y, but θ has a non-trivial value, between zero and $1/2$, in other cases. These results significantly expand the (up till now) rather small class of soluble first-passage problems with non-trivial exponents.

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